We can finally give the A.C. of $\zeta(s)$.

Thm 4-11 The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ ; \ \mathrm{Re}\, s > 1$$

extends to be a meromorphic function in $\mathbb{C}$. Its only pole is at $s=1$ which is simple. Moreover if $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ then $\Lambda(s)$ satisfies the func'l eqn $\Lambda(s) = \Lambda(1-s)$

Proof. We have seen that $\Lambda(s)$ has the integral repn

$$\Lambda(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) = \frac{1}{2}\int_0^{\infty}\left[\theta(it)-1\right] t^{s/2}\frac{dt}{t}.$$

which is valid for $\mathrm{Re}\, s>1$. We'll show that this integral repn can be extended to all $s$. To this end we first note

$$\frac{1}{2}\left(\theta(it)-1\right) = \sum_{n=1}^{\infty} e^{-\pi n^2 t} \leq \sum_{k=1}^{\infty} e^{-\pi k t}$$

$$= \frac{e^{-\pi t}}{1-e^{-\pi t}}$$

So for $t>1$ we have $\frac{1}{2}\left(\theta(it)-1\right) \leq e^{-\pi t}\cdot \underbrace{\frac{1}{1-e^{-\pi}}}_{=:K}.$

Next we divide the integral from $t=0$ to $t=\infty$ into 2 pieces

$$2\Lambda(s) = \int_0^1 \left(\theta(it)-1\right) t^{s/2}\frac{dt}{t} + \int_1^\infty (\cdots)\ \frac{dt}{t}.$$

Since for $t>1$, $\frac{1}{2}\left(\theta(it)-1\right) \leq K e^{-\pi t}$

the second integral in fact converges $\forall s \in \mathbb{C}$.

(not just $\operatorname{Re} s>1$)

For $\operatorname{Re} s>1$, the first integral

$$\frac{1}{2}\int_0^1 \left(\theta(it)-1\right) t^{s/2}\frac{dt}{t} = \frac{1}{2}\int_0^1 \theta(it)\, t^{s/2}\frac{dt}{t} - \frac{1}{2}\int_0^1 t^{s/2}\frac{dt}{t}$$

Since $\frac{1}{2}\int_0^1 t^{s/2}\frac{dt}{t} = +\frac{1}{s}$ we have that

for $\operatorname{Re} s>1$

$$\Lambda(s) = \frac{1}{2}\int_0^1 \theta(it)\, t^{s/2}\frac{dt}{t} + \frac{1}{s} + \frac{1}{2}\int_1^\infty \left(\theta(it)-1\right) t^{s/2}\frac{dt}{t}$$

For the first integral we'll use the fun'l eqn of $\theta$; $\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}}\sum e^{-\pi n^2/t}$ to get

$$\frac{1}{2}\int_0^1 \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\, t^{s/2}\frac{dt}{t} = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}\, t^{s/2}\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} - 1\right) t^{\frac{s}{2}-\frac{1}{2}}\frac{dt}{t} + \frac{1}{2}\int_0^1 t^{\frac{s}{2}-\frac{1}{2}}\frac{dt}{t}$$

$$= \frac{1}{2}\int_0^1 \left[\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} - 1\right] t^{\frac{s}{2}-\frac{1}{2}} \frac{dt}{t} + \frac{1}{s-1}$$

Now let $\tilde{t} = \frac{1}{t}$ to get

$$= \frac{1}{2}\int_1^\infty \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 \tilde{t}} - 1\right) \tilde{t}^{\frac{1}{2}(1-s)} \frac{d\tilde{t}}{\tilde{t}} + \frac{1}{s-1}$$

$$= \underbrace{\frac{1}{2}\int_1^\infty \left(\Theta(it) - 1\right) t^{\frac{1}{2}(1-s)} \frac{dt}{t}}_{\text{This integral again converges } \forall s\in\mathbb{C}} + \frac{1}{s-1}.$$

Putting everything together we have

$$\Lambda(s) = \frac{1}{2}\int_1^\infty \left(\Theta(it)-1\right) t^{\frac{1}{2}(1-s)} \frac{dt}{t} + \frac{1}{s-1}$$

$$+ \frac{1}{2}\int_1^\infty \left(\Theta(it)-1\right) t^{s/2} \frac{dt}{t} - \frac{1}{s}.$$

Now the RHS makes sense for every $s\in\mathbb{C}$ and gives the analytic continuation to $\Lambda(s)$ with simple poles at $s=1$ and $s=0$ w/ residues $1$ and $-1$ resp.

Since $\pi^{-s/2}\Gamma(s/2)$ is also meromorphic with simple poles at $s=0,-2,-4,\ldots$

We conclude that $\zeta(s)$ also has merom. continuation to all $s \in \mathbb{C}$ with only one simple pole at $s=1$.
Since $\Lambda(s)$ has no poles other than $s=0$, $s=1$, and $\Gamma(\frac{s}{2})$ has poles at $s=0,-2,\ldots$, $\zeta(s)$ must have zeroes at $s=-2,-4,\ldots$. These zeroes are called the trivial zeroes of $\zeta(s)$.

<u>Riemann hyp.</u> If $\zeta(s)=0$ and $s \notin \{-2,-4,\ldots\}$ then $\operatorname{Re} s = \frac{1}{2}$.

Finally one can show using fund. thm of arithmetic that $\zeta(s)$ has an Euler product expansion

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1} \quad \operatorname{Re} s > 1$$

$\downarrow$

Exercise

We can now also give the analog statement of A.C. and func'l eqn for L-functions of modular forms.

Recall if $f = \sum_{n=0}^{\infty} a_n n^{-s} \in S_k$ then

$a_n = O(n^{k/2})$ and if $f \in M_k$ then
$a_n = O(n^k)$

Hense $L(f,s) := \sum_{n=1}^{\infty} a_n n^{-s}$ converges

abs for $\operatorname{Re} s > c+1$ with either
$c = k/2$ or $c = k$ and defines
a holom function for $\operatorname{Re} s > c+1$.

<u>Thm 4.12</u> Let $f \in M_k$ then the L-function $L(f,s)$ has meromorphic continuation to all of $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(f,s) := (2\pi)^{-s} \Gamma(s) L(f,s) = (-1)^{k/2} \Lambda(f, k-s)$$

$\Lambda(f,s)$ is analytic $\forall s$ if $f \in S_k$
and if $f \in M_k \setminus S_k$ then $\Lambda(f,s)$ has simple poles at $s=0$ and $s=k$ with residues $-a_0$ and $+a_0(-1)^{k/2}$ resp.

<u>Proof</u> is very similar to the case of $\xi(s)$

We start by noting that $\Lambda(f,s)$ has integral repn $\Lambda(f,s) = \int_0^\infty (f(iy)-a_0)\, y^s \frac{dy}{y}$

Since $\int_0^\infty (f(iy)-a_0)\, y^s \frac{dy}{y} = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y}$

$$= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y} = \sum_{n=1}^\infty a_n (2\pi n)^{-s}\, \Gamma(s)$$

If $f \in S_k$ then $f(iy) \to 0$ exponentially fast as $y \to \infty$

$f(iy) = e^{-2\pi y}(a_1 + \dots) \to 0$ as $y \to \infty$.

$f(i/y) = (iy)^{k/2} f(iy)$ since $f$ is modular

i.e. $f(iy) = (-1)^{k/2} y^{-k} f(i/y)$

Hence $f(iy) \to 0$ as $y \to 0$ as well.

Hence the integral $\int_0^\infty f(iy)\, y^s \frac{dy}{y}$

in fact converges abs $\forall s$ and gives the analytic continuation of $\Lambda(f,s)$ to all of $s \in \mathbb{C}$.

To see the func'l eqn:

$$\Lambda(f,s) = \int_0^\infty f(iy) y^s \frac{dy}{y} = \int_0^\infty (-1)^{k/2} f(i/y)\, y^{-k+s} \frac{dy}{y}$$

$y \to 1/y$ $$= (-1)^{k/2} \int_0^\infty f(iy)\, y^{k-s} \frac{dy}{y} = (-1)^{k/2} \Lambda(f, k-s)$$

If $f$ is not a cusp form, ie $a_0 \neq 0$

Then for $\mathrm{Re}\, s > k$ we still have

$$(2\pi)^{s}\Gamma(s).L(f,s) = \int_0^\infty \left(f(iy)-a_0\right) y^s \frac{dy}{y} \qquad \mathrm{Re}\, s > k$$

$$= \int_0^1 \left(f(iy)-a_0\right) y^s \frac{dy}{y} + \underbrace{\int_1^\infty \underbrace{\left(f(iy)-a_0\right)}_{O(e^{-2\pi y})} y^s \frac{dy}{y}}_{\text{In fact converges } \forall s}$$

For $\mathrm{Re}\, s > k$

$$\Lambda(f,s) = \underbrace{-a_0 \int_0^1 y^s \frac{dy}{y}}_{-\frac{a_0}{s}} + \int_0^1 f(iy)\, y^s \frac{dy}{y} + \int_1^\infty \left(f(iy)-a_0\right) y^s \frac{dy}{y}$$

$y \to 1/y$
$$= -\frac{a_0}{s} + \int_1^\infty f(i/y)\, y^{-s} \frac{dy}{y} + \underbrace{\int_1^\infty \left(f(iy)-a_0\right) y^s \frac{dy}{y}}$$

$$= -\frac{a_0}{s} + \int_1^\infty (-1)^{k/2} y^{k-s} f(iy) \frac{dy}{y} + \quad ''$$

$$= -\frac{a_0}{s} + (-1)^{k/2} \int_1^\infty \left(f(iy) - a_0\right) y^{k-s} \frac{dy}{y} + (-1)^{k/2} \int_1^\infty a_0\, y^{k-s} \frac{dy}{y}$$

$$+ \int_1^\infty \left(f(iy)-a_0\right) y^s \frac{dy}{y} \qquad \mathrm{Re}\, s > k$$

$$= -\frac{a_0}{s} + \underbrace{\int_1^\infty \left(f(iy)-a_0\right)\left((-1)^{k/2} y^{k-s} + y^s\right)\frac{dy}{y}}_{\text{converges } \forall s.} - (-1)^{k/2} \frac{a_0}{k-s}$$

RHS gives the meromorphic continuation of $\Lambda(f,s)$ to all of $s \in \mathbb{C}$ w/ simple poles at $s=0$ and $s=k$.
Note that RHS has a symmetry as $s \to k-s$ which gives the func'l eqn

$$\Lambda(f,s) = (-1)^{k/2} \Lambda(f, k-s)$$

Rmk ① If $k \equiv 2 \mod 4$ then $(-1)^{k/2} = -1$

and $\Lambda(f,s) = -\Lambda(f, k-s)$

This gives a trivial vanishing of $\Lambda(f,s)$ at $s = k/2$.

② We've seen that given $f \in M_k$ its L-function has A.C. and F.E.

Natural question Is there a converse?
Ie if we have a Dirichlet series $D(s) = \sum a_n n^{-s}$ formed from a sequence of numbers $a_n = O(n^\alpha)$, and $D(s)$ has A.C. and a F.E. of the above type can we conclude that $a_n$'s are F. coefs of some $f \in M_k$?
Hecke (Weil for congruence s/gps) showed that the answer is "essentially" yes w/ some qualifications

<u>Thm</u> (Hecke) <u>Converse thm</u>

Let $f(z)$, $g(z)$ be holomorphic functions on $\mathbb{H}$, $f(z) = \sum_{n=0}^{\infty} a_n q^n$, $g(z) = \sum b_n q^n$ which converges abs and unif on compact ssets of $\mathbb{H}$. Furthermore assume $\exists\ \alpha \geq 0$ s.t $a_n = O(n^\alpha)$, $b_n = O(n^\alpha)$

Then for positive $k, N \in \mathbb{Z}$ the following 2 conditions (A) and (B) are equivalent

(A) $g(z) = (i\sqrt{N} z)^{-k} f(-1/Nz) = \left(f \Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\right)(z)$

(B) Both $\Lambda_N(f,s)$ and $\Lambda_N(g,s)$ can be continued analytically to the whole $s$ plane and satisfy the Func eqn

$$\Lambda_N(f,s) = \Lambda_N(g, k-s)$$

and $\Lambda_N(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s}$ is holom $\forall s \in \mathbb{C}$

and bounded on any vertical strip $\{s \in \mathbb{C} :$ $\beta < \mathrm{Re}\, s$

Here $\Lambda_N(f,s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}$.

<u>For a proof see for example Miyake's book</u>

The next natural question is the following.

We've seen that $L(f,s)$ for an $f \in \mathcal{M}_k$ has A.C. and func'l eqn as the Riemann zeta function $\zeta(s)$

$\zeta(s)$ also has an Euler product $\prod(1-p^{-s})^{-1}$

Ques. Does $L(f,s)$ also have an Euler product

Ans In general NO!

But we'll see that $\mathcal{M}_k$ has a basis of forms $f_1, \ldots f_d$ s.t for each $f_i$, $L(f_i, s)$ can also be written as an infinite product over primes.

To prove this result we need to introduce Hecke operators.

Before we do that we'll introduce one more function whose defn is similar to Eisenstein series (ie it is an average over the group) but this function will be also a cusp form.

# §5 Poincaré series

The idea goes back to Poincaré.

Let $\{\mu_\gamma \mid \gamma \in \Gamma\}$ be a collection of holomorphic, no where vanishing functions on $\mathbb{H}$ satisfying the "automorphy" condition

$$\mu_{\alpha\beta}(z) = \mu_\alpha(\beta z)\mu_\beta(z) \quad \forall\ \alpha, \beta \in \Gamma$$

e.g $\mu_\gamma(z) = (cz+d)^k$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such a function.

We want to construct a holomorphic function $f(z)$ on $\mathbb{H}$ s.t $f(\gamma z) = \mu_\gamma(z) f(z)\ \forall \gamma \in \Gamma$

Let $h(z)$ be any holom function on $\mathbb{H}$ we write formally $f(z) = \sum_{\gamma \in \Gamma} \frac{h(\gamma z)}{\mu_\gamma(z)}$

Then $f(\alpha z) = \sum_{\gamma \in \Gamma} \frac{h(\gamma \alpha z)}{\mu_\gamma(\alpha z)} = \sum_\gamma \frac{h(\gamma\alpha z)}{\mu_{\gamma\alpha}(z)} \mu_\alpha(z)$

$$= \mu_\alpha(z) \sum_{\gamma \in \Gamma} \frac{h(\gamma\alpha z)}{\mu_{\gamma\alpha}(z)} = \mu_\alpha(z) f(z)$$

Since as $\gamma$ runs over $\Gamma$, so does $\gamma\alpha$ for fixed $\alpha$.

If the series defining $f$ converges abs, and unif on compact ssets of $\mathbb{H}$ then $f(z)$ is holomorphic and the formal calculation becomes legitimate

But there could be $\infty$ly many $\gamma$'s for which for example $\mu_\gamma(z) =$ a constant say $1$ then we have little hope of convergence

Let $\Gamma_\infty = \{\gamma \in \Gamma \mid \mu_\gamma(z) \equiv 1\}$

Check this is a s/gp of $\Gamma$. It also follows from the automorphy condition on $\mu$.

Let $\Gamma = \bigcup_{\gamma \in R} \Gamma_\infty \gamma$ $\quad R$ is a set of reps for $\Gamma_\infty \backslash \Gamma$

Now choose $h(z)$ so that $h(\gamma z) = h(z)$ $\forall \gamma \in \Gamma_\infty$, and define

$$f(z) = \sum_{\gamma \in R} \frac{h(\gamma z)}{\mu_\gamma(z)}$$

(1) First of all $f$ is well defined, i.e. indep of choice of reps for $\Gamma_\infty \backslash \Gamma$

If $\gamma, \tilde{\gamma}$ are in the same coset mod $\Gamma_\infty$ then $\gamma = \beta \tilde{\gamma}$ for some $\beta \in \Gamma_\infty$

$$h(\gamma z) = h(\beta \tilde{\gamma} z) = h(\beta(\tilde{\gamma} z)) = h(\tilde{\gamma} z)$$

since $\beta \in \Gamma_\infty$ and $h(\beta z) = h(z)$ $\forall \beta \in \Gamma_\infty$.

$$\mu_\gamma(z) = \mu_{\beta\tilde{\gamma}}(z) = \underbrace{\mu_\beta(\tilde{\gamma} z)}_{\equiv 1 \text{ since } \beta \in \Gamma_\infty} \mu_{\tilde{\gamma}}(z) = \mu_{\tilde{\gamma}}(z)$$

Hence $f$ is well defined.

As before one can show that

$$f(\alpha z) = \mu_\alpha(z) f(z)$$

using the fact that if $R$ is a set of reps so is $R\alpha$ for fixed $\alpha \in \Gamma$

$$\Gamma = \bigcup_{\gamma \in R} \Gamma_\infty \gamma_R \qquad \Gamma = \Gamma\alpha = \bigcup \Gamma_\infty (\gamma_R \alpha)$$

Now in the special case that

$$\mu_\gamma(z) = (cz+d)^k$$

$$\Gamma_\infty := \{\gamma \in \Gamma \mid (cz+d)^k \equiv 1\} \qquad k = \text{even}$$

$$= \left\{ \pm \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\},$$

and $R = \left\{ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \geq 0 \right\}$,

$$= \{ (c,d) \in \mathbb{Z}^2 \mid c \geq 0, (c,d) = 1 \}$$

If we take $h(z) \equiv 1$ we get the Eisenstein series as before

on the other $e^{2\pi i m z}$ for $m \in \mathbb{N}$ is also invariant under $\Gamma_\infty$ and we define

<u>Defn</u> The m-th <u>Poincare series</u> of wt $k$ for $\Gamma$ is the function

$$P_m(z) = P_m^k(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-k} \underbrace{e^{2\pi i m \gamma z}}_{:= e(m\gamma z)}$$